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On the Rate of Convergence of Bernstein Power Series for Functions of Bounded Variation

MOHAMMAD KAZIM KHAN

Department of Mathematics, Kent State University, Kent, Ohio 44242, U.S.A.

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1. INTRODUCTION

Let $L_j^{(\alpha)}$ $(\alpha > -1)$ denote the Laguerre polynomials of degree j where α is a parameter. For $f \in C[0, 1]$, consider the Bernstein power series operator [2]

$$P_n(f, v) = (1-v)^{n+1} \exp\left(\frac{\omega v}{1-v}\right) \sum_{j=0}^{\infty} f\left(\frac{j}{j+n}\right) L_j^{(n)}(\omega)\, v^j; \qquad 0 \leqslant v \leqslant a < 1, \tag{1.1}$$

where $\omega \leqslant 0$ is a parameter. Cheney and Sharma [2] proved that, for $v \in [0, a]$,

$$P_n(f, v) \to f(v), \qquad \text{as } n \to \infty \text{ and } \frac{\omega}{n} \to 0, \tag{1.2}$$

and the convergence is uniform. Khan [5] provided a probabilistic proof of (1.2) by using weak convergence of probability measures. For $\omega = 0$, (1.1) reduces to the modified Meyer–König Zeller operator:

$$M_n(f, v) = (1-v)^{n+1} \sum_{j=0}^{\infty} f\left(\frac{j}{j+n}\right) \binom{n+j}{j} v^j. \tag{1.3}$$

Cheney and Sharma [2] proved that if f is convex then

$$M_n(f, v) \geqslant M_{n+1}(f, v) \tag{1.4}$$

for all n. By using properties of conditional expectations of negative

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binomial distribution, Khan [5] provided a simple probabilistic proof of (1.4). The Meyer–König Zeller operator (unmodified) is defined as

$$K_n(f, v) = (1-v)^{n+1} \sum_{j=0}^{\infty} f\left(\frac{j}{j+n+1}\right) \binom{n+j}{j} v^j. \tag{1.5}$$

Convergence properties of $K_n(f, v)$ are available in [9, pp. 164–169] for differentiable functions. $K_n(f, v)$ is analogously extended to

$$Z_n(f, v) = (1-v)^{n+1} \exp\left(\frac{\omega v}{1-v}\right) \sum_{j=0}^{\infty} f\left(\frac{j}{j+n+1}\right) L_j^{(n)}(\omega)\, v^j, \tag{1.6}$$

$0 \leqslant v \leqslant a < 1$, where $\omega \leqslant 0$ is a parameter.

In this paper, the rates of convergence of (1.1) and (1.6) are obtained for both continuous functions and discontinuous functions of bounded variation. Furthermore, it is shown that (1.4) type inequality holds for (1.5) when f is continuous and convex.

2. An Extension

Many classical operators can be simultaneously studied by the following operator. Let $\{X_{j,n}; j = 1, 2, ..., n; n \geqslant 1\}$ be a triangular array of independent random variables such that for each fixed n, $X_{1,n}, X_{2,n}, ..., X_{n,n}$ are identically distributed with $E(X_{1,n}) = \mu_n(x)$ and finite variance $\mathrm{Var}(X_{1,n}) = \sigma_n^2(x) > 0$, where $x \in I \subseteq \mathbb{R}$ is a real parameter. Define $S_n = X_{1,n} + \cdots + X_{n,n}$. Let h be a well-defined measurable function on $\mathbb{R}$ and let $\{a_n\}$ be a sequence of positive numbers. Define an approximation operator by

$$A_n(h, x) = E\{h(a_n S_n)\} = \int_{-\infty}^{\infty} h(a_n u)\, dF_{n,x}(u) \tag{2.1}$$

when $E|h(a_n S_n)| < \infty$, where $F_{n,x}(u)$ is the distribution function (d.f.) of S_n. If $X_{j,n}$, $j = 1, 2, ..., n$, are identically distributed for all n, $\mu_n(x) = x$, $\sigma_n^2(x) = \sigma^2(x) > 0$, and $a_n = 1/n$ then $A_n(h, x)$ reduces to the Feller operator [3]. Khan [5, 6] studied the properties of the Feller operator for $h \in C(\mathbb{R})$. Khan [4] provided the rate of convergence of the Feller operator for $h \in BV(\mathbb{R})$.

It is simple to verify (see [5] for example) that a number of classical operators, such as Bernstein, Szász, Baskakov, Gamma, and Weirstrass operators, are special cases of the Feller operator. However, to study (1.1) and (1.6) one needs the extension provided in (2.1).

To show that (1.1) and (1.6) are special cases of (2.1), we proceed as follows. Let $X_{0,n}, X_{1,n}, \ldots, X_{n,n}$ be a sequence of independent and identically distributed triangular arrays of random variables with the probability mass function (which we will refer to as the Laguerre distribution)

$$P(X_{0,n}=j)=(1-v)\exp\left(\frac{\omega^* v}{1-v}\right) L_j^{(0)}(\omega^*)\, v^j$$

for $j=0, 1, 2, \ldots, 0 \leqslant v \leqslant a < 1, \omega^* = \omega/(n+1), \omega \leqslant 0$. The generating function of $X_{0,n}$ is

$$\varphi_{X_{0,n}}(s)=(1-v)\exp\left(\frac{\omega^* v}{1-v}\right)(1-sv)^{-1}\exp\left(\frac{-\omega^* sv}{1-sv}\right).$$

The generating function of $S_{n+1}=X_{0,n}+X_{1,n}+\cdots+X_{n,n}$ is obtained by the convolution property as

$$\varphi_{S_{n+1}}(s)=(1-v)^{n+1}\exp\left(\frac{\omega v}{1-v}\right)(1-sv)^{-(n+1)}\exp\left(\frac{-\omega sv}{1-sv}\right).$$

By comparing $\varphi_{S_{n+1}}(s)$ with the generating function of Laguerre polynomials [8], the probability mass function of S_{n+1} is obtained as

$$P(S_{n+1}=j)=(1-v)^{n+1}\exp\left(\frac{\omega v}{1-v}\right) L_j^{(n)}(\omega)\, v^j, \tag{2.2}$$

$j=0, 1, 2, \ldots, 0 \leqslant v \leqslant a < 1$, $\omega \leqslant 0$. Clearly, for $f \in C[0, 1]$ and $v \in [0, a]$, $a < 1$,

$$P_n(f, v)=E\left\{f\left(\frac{S_{n+1}}{n+S_{n+1}}\right)\right\}=E\{h(a_{n+1}S_{n+1})\},$$

where $a_{n+1}=1/n$, $h(t)=f(r(t))$, $r(t)=t/(1+t)$, $t \geqslant 0$. Hence, $P_n(f, v)=A_{n+1}(h, x)$ where $x=v/(1-v)$. Similarly, if we take $a_{n+1}=(n+1)^{-1}$ then $Z_n(f, v)=A_{n+1}(h, x)$.

The following identities concerning Laguerre distribution will be needed in the sequel,

$$E(X_{0,n})=-\frac{v^2-(1-\omega^*)\,v}{(v-1)^2} \tag{2.3}$$

$$\operatorname{Var}(X_{0,n})=\frac{(1+\omega^*)\,v^2+(\omega^*-1)\,v}{(v-1)^3} \tag{2.4}$$

$$E(X_{0,n})^3=\frac{-\tau_1-\tau_2}{(v-1)^6}, \tag{2.5}$$

where

$$\tau_1 = v^6 + (7\omega^* + 1)\, v^5 + (6\omega^{*2} - 4\omega^* - 8)\, v^4$$

$$\tau_2 = (\omega^{*3} - 3\omega^{*2} - 12\omega^* + 8)\, v^3 + (-3\omega^{*2} + 8\omega^* - 1)\, v^2 + (\omega^* - 1)\, v.$$

3. Main Results (Continuous Case)

THEOREM 1. *Let $A_n(h, x)$ be as defined by* (2.1) *and $h \in C(I)$. Then for* $n = 1, 2, \ldots$

$$|A_n(h, x) - h(x)| \leqslant (1 + K_n(x))\, \omega\left(h, \frac{1}{\sqrt{n}}\right),$$

where $\omega(h, \delta)$ is the modulus of continuity of h and

$$K_n(x) = (na_n \sigma_n(x))^2 + n(na_n \mu_n(x) - x)^2. \tag{3.1}$$

Proof. For $\delta > 0$,

$$|h(y) - h(x)| \leqslant (1 + [\,|y - x|/\delta\,])\, \omega(h, \delta),$$

where $[z]$ is the greatest integer $\leqslant z$. Therefore,

$$\begin{aligned} |A_n(h, x) - h(x)| &\leqslant E|h(a_n S_n) - h(x)| \\ &\leqslant (1 + E[\,|a_n S_n - x|/\delta\,])\, \omega(h, \delta) \\ &\leqslant (1 + E(a_n S_n - x)^2/\delta^2)\, \omega(h, \delta) \\ &= (1 + \{a_n^2 \operatorname{Var}(S_n) + (na_n \mu_n(x) - x)^2\}/\delta^2)\, \omega(h, \delta). \end{aligned}$$

By taking $\delta = 1/\sqrt{n}$, the result follows.

LEMMA 1. *Let $X_{1,n}, \ldots, X_{n,n}$ be a sequence of independent and identically distributed (for all n) non-negative random variables with finite expectation. Let $\{a_n\}$ be a sequence of positive numbers such that $\{na_n\}$ is non-increasing. Then*

$$E\{a_{n-1} S_{n-1} \mid S_n\} \overset{\text{a.s.}}{\geqslant} a_n S_n.$$

Equality holds if $\{na_n\}$ is constant.

Proof. It is well known [5] that

$$E\left(\frac{S_{n-1}}{n-1} \,\middle|\, S_n\right) \overset{\text{a.s.}}{=} \frac{S_n}{n}.$$

Therefore, if $\{na_n\}$ is constant, then the lemma follows. Otherwise,

$$E(a_{n-1}S_{n-1} \mid S_n) \overset{\text{a.s.}}{=} a_{n-1}E(S_{n-1} \mid S_n)$$
$$\overset{\text{a.s.}}{=} a_{n-1}\left(\frac{n-1}{n}\right)S_n$$
$$\overset{\text{a.s.}}{\geqslant} a_n S_n$$

since $\{na_n\}$ is non-increasing.

THEOREM 2. *Let $A_n(h, x)$ be as defined in (2.1) where $X_{j,n}$ are non-negative random variables identically distributed for all n. Let h be non-decreasing and convex and $\{na_n\}$ be a non-increasing sequence of positive numbers. Then for $n = 2, 3, \ldots$*

$$A_{n-1}(h, x) \geqslant A_n(h, x).$$

Proof.

$$\begin{aligned} A_{n-1}(h, x) &= E(h(a_{n-1}S_{n-1})) \\ &= E\{E(h(a_{n-1}S_{n-1}) \mid S_n)\} \\ &\geqslant E\{h(E(a_{n-1}S_{n-1} \mid S_n))\} && \text{Jensen's inequality} \\ &\geqslant E\{h(a_nS_n)\} && \text{monotonicity and Lemma 1} \\ &= A_n(h, x). \end{aligned}$$

Remark. h need not be non-decreasing in Theorem 2 if $a_n = 1/n$ [5].

4. MAIN RESULTS (DISCONTINUOUS CASE)

In this section, unless otherwise stated, it will be assumed that h is a normalized function of bounded variation. Also, it will be assumed that $E|X_{1,n}^3| < \infty$ for $n = 1, 2, \ldots$. The results obtained in this section extend the results in [4]. The following theorem, an extension of the Barry–Esseen bound for the central limit Theorem [7, pp. 62], will be needed in the sequel.

THEOREM 3. *Let $\{Y_{j,n}, j = 1, 2, \ldots, n\}$ be a triangular array of random variables. Suppose that for each n the random variables $Y_{1,n}, \ldots, Y_{n,n}$ are independent with zero means, and are normalised so that their variances add up to one:*

$$\sum_{j=1}^{n} E(Y_{j,n}^2) = 1, \qquad n \geqslant 1.$$

Also, let $\max_{1 \leqslant j \leqslant n} E(Y_{j,n}^2) \to 0$ *as* $n \to \infty$ *and let* $T_n = \sum_{j=1}^{n} Y_{j,n}$. *Assume that Lindeberg's condition holds,*

$$\sum_{j=1}^{n} E(Y_{j,k}^2 I(|Y_{j,n}| > \varepsilon)) \to 0 \qquad \text{as} \quad n \to \infty \text{ for all } \varepsilon > 0,$$

where I *is the characteristic function. Then there exists a numerical constant* $c < \infty$ *such that for all* z *and all* n, *if* F_n^* *is the* d.f. *of* T_n *and* G^* *is the* d.f. *of a standard normal random variable, then*

$$|F_n^*(z) - G^*(z)| \leqslant c \sum_{j=1}^{n} E|Y_{j,n}|^3.$$

Here

$$G^*(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

In our case, namely (2.1), $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ are identically distributed. Define

$$Y_{j,n} = \frac{X_{j,n} - \mu_n(x)}{\sqrt{n}\,\sigma_n(x)}.$$

Clearly, $\max_{1 \leqslant j \leqslant n} E(Y_{j,n}^2) = 1/n \to 0$ as $n \to \infty$. Furthermore, Lindeberg's condition would hold if

$$\lim_{n \to \infty} \frac{E|X_{1,n} - \mu_n(x)|^3}{\sqrt{n}\,\sigma_n^3(x)} = 0.$$

For Bernstein power series, namely (1.1) and (1.6), $E|X_{1,n} - \mu_n(x)|^3 = O(1)$ and $\sigma_n^2(x) \to x(1-x)^{-2}$. Therefore Lindeberg's condition would hold. Hence, Theorem 3 simplifies to

$$|F_{n,x}^*(z) - G^*(z)| \leqslant \frac{cE|X_{1,n} - \mu_n(x)|^2}{\sqrt{n}\,\sigma_n^3(x)},$$

where

$$F_{n,x}^*(z) = P\left(\sqrt{n}\left(\frac{S_n}{n} - \mu_n(x)\right)\Big/\sigma_n(x) \leqslant z\right).$$

THEOREM 4. *Let* $h \in BV(-\infty, \infty)$, *and*

$$\lim_{n \to \infty} \frac{E|X_{1,n} - \mu_n(x)|^3}{\sqrt{n}\,\sigma_n^3(x)} = 0.$$

Then for every $x \in (-\infty, \infty)$ *and all* $n = 1, 2, \ldots,$ *for the operator* (2.1) *we have*

$$|A_n(h, x) - \bar{h}(x)| \leqslant \frac{P_n(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{Q_n(x)}{\sqrt{n}} \tilde{h}(x),$$

where $I_k = [x - 1/\sqrt{k}, x + 1/\sqrt{k}]$, $k = 1, 2, \ldots, n$, $I_0 = (-\infty, \infty)$. $P_n(x) = 2K_n(x) + 1$, *where* $K_n(x)$ *is provided in* (3.1),

$$Q_n(x) = \frac{5c\, E|X_{1,n} - \mu_n(x)|^3}{2\sigma_n^3(x)} + \frac{|na_n \mu_n(x) - x|}{\sqrt{2\pi}\, a_n \sigma_n(x)},$$

$\tilde{h}(x) = |h(x^+) - h(x^-)|$, $\bar{h}(x) = (h(x^+) + h(x^-))/2$, *c is given in Theorem* 3, *and*

$$g_x(t) = \begin{cases} h(t) - h(x^+) & \text{if } t > x \\ 0 & \text{if } t = x \\ h(t) - h(x^-) & \text{if } t < x. \end{cases}$$

Proof. Note that since h is normalized,

$$h(t) = g_x(t) + \bar{h}(t) + \frac{h(x^+) - h(x^-)}{2} \operatorname{sgn}_x(t)$$

where

$$\operatorname{sgn}_x(t) = \begin{cases} -1 & \text{if } t > x \\ 0 & \text{if } t = x \\ 1 & \text{if } t < x. \end{cases}$$

Hence,

$$|A_n(h, x) - \bar{h}(x)| \leqslant |A_n(g_x, x)| + \tfrac{1}{2} \tilde{h}(x) |A_n(\operatorname{sgn}_x, x)|.$$

First consider

$$\begin{aligned} A_n(\operatorname{sgn}_x, x) &= P(a_n S_n > x) - P(a_n S_n < x) \\ &= 1 - 2P(a_n S_n \leqslant x) + P(a_n S_n = x) \\ &= 2(\tfrac{1}{2} - F^*_{n,x}(t_n)) + F^*_{n,x}(t_n^+) - F^*_{n,x}(t_n^-), \end{aligned}$$

where $t_n = (x - na_n \mu_n(x))/(a_n \sigma_n(x) \sqrt{n})$. Therefore,

$$|A_n(\operatorname{sgn}_x, x)| \leqslant 2|F^*_{n,x}(t_n) - G^*(0)| + |F^*_{n,x}(t_n) - F^*_{n,x}(t_n^-)|.$$

Now

$$|A_n(\operatorname{sgn}_x, x)| \leqslant 2\{|F^*_{n,x}(t_n) - G^*(t_n)| + |G^*(t_n) - G^*(0)|\}$$
$$+ |F^*_{n,x}(t_n) - G^*(t_n)| + |G^*(t_n) - F^*_{n,x}(t_n^-)|.$$

Note that for any ε_n,

$$|F^*_{n,x}(t_n^-) - G^*(t_n)| \leqslant \max\{|F^*_{n,x}(t_n + \varepsilon_n) - G^*(t_n)|, |F^*_{n,x}(t_n - \varepsilon_n) - G^*(t_n)|\}.$$

Now

$$|F^*_{n,x}(t_n \pm \varepsilon_n) - G^*(t_n)| \leqslant |F^*_{n,x}(t_n \pm \varepsilon_n) - G^*(t_n \pm \varepsilon_n)|$$
$$+ |G^*(t_n \pm \varepsilon_n) - G^*(t_n)|.$$

It is easy to verify that for all $z \in (-\infty, \infty)$,

$$|G^*(z \pm \varepsilon_n) - G^*(z)| \leqslant \frac{1}{\sqrt{2\pi}} |\varepsilon_n|.$$

Therefore, by Theorem 3,

$$|A_n(\operatorname{sgn}_x, x)| \leqslant 2\left\{\frac{cE|X_{1,n} - \mu_n(x)|^3}{\sqrt{n}\,\sigma_n^3(x)} + \frac{|t_n|}{\sqrt{2\pi}}\right\}$$
$$+ \frac{2cE|X_{1,n} - \mu_n(x)|^3}{\sqrt{n}\,\sigma_n^3(x)} + \frac{|\varepsilon_n|}{\sqrt{2\pi}}.$$

Take $\varepsilon_n = c\sqrt{2\pi}E|X_{1,n} - \mu_n(x)|^3/(\sqrt{n}\,\sigma_n^3(x))$. Then,

$$|A_n(\operatorname{sgn}_x, x)| \leqslant \frac{2}{\sqrt{n}}\left\{\frac{5cE|X_{1,n} - \mu_n(x)|^3}{2\sigma_n^3(x)} + \frac{|x - na_n\mu_n(x)|}{\sqrt{2\pi}\,a_n\sigma_n(x)}\right\}$$
$$= \frac{2}{\sqrt{n}} Q_n(x).$$

Now consider $A_n(g_x, x)$, $g_x \in BV(-\infty, \infty)$, $g_x(x) = 0$. For simplicity of notation $V_{[a,b]}(g_x) = V_{[a,b]}$ will denote the total variation of g_x on $[a, b]$. The following technique is due to Bojanic and Vuilleumier [1],

$$A_n(g_x, x) = \left(\int_{-\infty}^{\alpha} + \int_{(\alpha,\beta)} + \int_{\beta}^{\infty}\right) g_x(t)\, d\bar{F}_{n,x}(t),$$

where $\bar{F}_{n,x}(t) = P(a_n S_n \leqslant t)$, $\alpha = x - 1/\sqrt{n}$, and $\beta = x + 1/\sqrt{n}$;

$$\left|\int_{(\alpha,\beta)} g_x(t)\, d\bar{F}_{n,x}(t)\right| \leqslant \int_{(\alpha,\beta)} |g_x(t) - g_x(x)|\, d\bar{F}_{n,x}(t) \leqslant V_{I_n}. \tag{4.1}$$

Integrating by parts, one gets

$$\int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t) = g_x(\alpha^+)\, \bar{F}_{n,x}(\alpha) + \int_{-\infty}^{\alpha} \hat{\bar{F}}_{n,x}(t)\, d(-g_x(t)),$$

where $\hat{\bar{F}}_{n,x}(t)$ is the normalized form of $\bar{F}_{n,x}(t)$ and $t \leqslant \alpha < x$, $|d(-g_x(t))| \leqslant d_t(-V_{[t,x]})$. Also

$$\begin{aligned}\hat{\bar{F}}_{n,x}(t) \leqslant \bar{F}_{n,x}(t) &= P(a_n S_n \leqslant t)\\ &\leqslant P(|a_n S_n - x| \geqslant |t-x|)\\ &\leqslant \frac{E(a_n S_n - x)^2}{(t-x)^2} \qquad \text{by Chebyshev's inequality.}\end{aligned}$$

Let $\xi_n(x) = E(a_n S_n - x)^2 = n a_n^2 \sigma_n^2(x) + (x - n a_n \mu_n)^2$. Therefore, $\bar{F}_{n,x}(\alpha) \leqslant n\xi_n(x)$. Hence,

$$\left| \int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t) \right| \leqslant n V_{I_n} \xi_n(x) + \xi_n(x) \int_{-\infty}^{\alpha} \frac{1}{(x-t)^2}\, d_t(-V_{[t,x]}).$$

Integrating by parts and a change of variable, one gets

$$\int_{-\infty}^{\alpha} \frac{1}{(x-t)^2}\, d_t(-V_{[t,x]}) \leqslant \int_0^n V_{[x - t^{-1/2},\, x]}\, dt \leqslant \sum_{k=0}^{n} V_{[x-k^{-1/2},\, x]},$$

where for $k=0$, $[x-k^{-1/2}, x] = (-\infty, x]$. Hence,

$$\left| \int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t) \right| \leqslant 2\xi_n(x) \sum_{k=0}^{n} V_{[x-k^{-1/2},\, x]}. \tag{4.2}$$

Similarly, by using the fact that $\bar{S}_{n,x}(t) = 1 - \bar{F}_{n,x}(t) = P(a_n S_n > t)$ is the left continuous, non-increasing survival function for the random variable $a_n S_n$, one gets

$$\left| \int_{\beta}^{\infty} g_x(t)\, d\bar{F}_{n,x}(t) \right| \leqslant 2\xi_n(x) \sum_{k=0}^{n} V_{[x,\, x+k^{-1/2}]}. \tag{4.3}$$

Combining (4.1), (4.2), and (4.3), we get

$$|A_n(g_x, x)| \leqslant \frac{2n\xi_n(x)+1}{n} \sum_{k=0}^{n} V_{I_k}.$$

By letting $P_n(x) = 2n\xi_n(x) + 1$, the theorem follows.

5. Special Cases

In the following, a few examples are provided. The emphasis is to show the versatility of Theorems 1, 2, and 4 and to provide explicit expressions for $K_n(x)$, $P_n(x)$, and $Q_n(x)$ in each case.

5.1. *Feller Operator*

Let $X_{1,n}$, $X_{2,n}, ..., X_{n,n}$ be independent and identically distributed random variables for all $n = 1, 2, ...$ (i.e., the d.f. of $X_{1,n}$ does not depend on n) such that $E(X_{1,n}) = \mu_n(x) = x$, $\text{Var}(X_{1,n}) = \sigma_n^2(x) = \sigma^2(x)$. Let $a_n = 1/n$. Then

$$A_n(h, x) = L_n(h, x) = \int_{-\infty}^{\infty} h\left(\frac{u}{n}\right) dF_{n,x}(u).$$

$L_n(h, x)$ is known as the Feller operator (cf. [5]). Now $K_n(x) = \sigma^2(x)$ and Theorem 1 specializes to

$$|L_n(h, x) - h(x)| \leqslant (1 + \sigma^2(x))\, \omega\left(h, \frac{1}{\sqrt{n}}\right). \tag{5.1}$$

This result is given in [5], Since $na_n = 1$, Theorem 2 provides that

$$L_{n-1}(h, x) \geqslant L_n(h, x)$$

for convex functions h. This result is proved in [5] as well. By Theorem 4, we get

$$|L_n(h, x) - \bar{h}(x)| \leqslant \frac{2\sigma^2(x) + 1}{n} \sum_{k=0}^{n} V_{I_k} + \frac{5cE|X_{1,1} - x|^3}{\sqrt{n}\, 2\sigma^3(x)} \tilde{h}(x).$$

This result is provided in [4]. For specific examples for Bernstein, Szász, Baskakov, Gamma, and Weirstrass operators see [4, 5, 6].

5.2. *Bernstein Power Series (Modified)*

As shown in Section 2, $P_n(f, v) = A_{n+1}(h, x)$ where $a_{n+1} = 1/n$, $h(t) = f(r(t))$, $r(t) = t/(1+t)$, $t \geqslant 0$, and $x = v/(1-v)$. Now $X_{0,n}, X_{1,n}, ..., X_{n,n}$ are independent and identically distributed random variables with

$$\begin{aligned} P(X_{0,n} = j) &= (1 - v) \exp\left(\omega^* \frac{v}{1-v}\right) L_j^{(0)}(\omega^*)\, v^j \\ &= \frac{1}{1+x} \exp(\omega^* x)\, L_j^{(0)}(\omega^*) \left(\frac{x}{1+x}\right)^j, \end{aligned}$$

$j=0, 1, 2, \ldots, \omega^*=\omega/(n+1)$, and $\omega \leqslant 0$. From (2.3), (2.4), and (2.5),

$$E(X_{0,n})=\mu_n(x)=-\frac{v^2-(1-\omega^*)\,v}{(v-1)^2}=x-x(1+x)\,\omega^*$$
$$\to x \quad \text{as} \quad n\to\infty$$

$$\sigma_n^2(x)=\operatorname{Var}(X_{0,n})=\frac{(1+\omega^*)\,v^2+(\omega^*-1)\,v}{(v-1)^3}=x(1+x)(1-\omega^*-2\omega^*x)$$
$$\to x(1+x)=\sigma^2(x) \quad \text{as} \quad n\to\infty$$

$$\begin{aligned}E(X_{0,n})^3=\lambda_n(x)= {} & -x(x^5+(7\omega^*+1)\,x^4(x+1)\\ & +(6\omega^{*2}-4\omega^*-8)\,x^3(x+1)^2\\ & +(\omega^{*3}-3\omega^{*2}-12\omega^*+8)\,x^2(x+1)^3\\ & +(-3\omega^{*2}+8\omega^*-1)\,x(x+1)^4\\ & +(\omega^*-1)(x+1)^5).\end{aligned}$$

Hence,

$$\begin{aligned}K_{n+1}(x)= {} & \left(\frac{n+1}{n}\right)^2 x(1+x)(1-\omega^*-2\omega^*x)\\ & +(n+1)\left(\frac{n+1}{n}(x-x(1+x)\,\omega^*)-x\right)^2.\end{aligned}$$

Clearly $\lim_{n\to\infty} K_{n+1}(x)=x(1+x)$. By Theorem 1,

$$|P_n(f, v)-f(v)|=|A_{n+1}(h, x)-h(x)|\leqslant(1+K_{n+1}(x))\,\omega\left(h, \frac{1}{\sqrt{n}}\right).$$

Since $\omega(h, \delta)\leqslant\omega(f, \delta)$, by Theorem 1 one gets

$$|P_n(f, v)-f(v)|\leqslant\left(1+K_{n+1}\left(\frac{v}{1-v}\right)\right)\omega\left(f, \frac{1}{\sqrt{n}}\right).$$

Note that if $0<v\leqslant a<1$, then the convergence is uniform. Also, if f is increasing and convex on $[0, 1]$ then h is increasing and convex on $[0, \infty)$. Since $\{na_n\}$ is non-increasing, by Theorem 2, for increasing convex functions f, $n=3, 4, \ldots,$

$$M_{n-2}(f, v)=A_{n-1}(h, x)\geqslant A_n(h, x)=M_{n-1}(f, v).$$

To apply Theorem 4, first note that if $f \in BV[0, 1]$ then $h \in BV[0, \infty)$. Extend $h(t) = h(0)$ for $t < 0$. From (2.3), (2.4), and (2.5) we get

$$E|X_{0,n} - \mu_n(x)|^3 \leqslant \rho_n(x) = \lambda_n(x) + 3\mu_n(x)\,\sigma_n^2(x) + 7\mu_n^3(x).$$

Hence,

$$|A_{n+1}(h, x) - \bar{h}(x)| \leqslant \frac{2K_{n+1}(x) + 1}{n} \sum_{k=0}^{n+1} V_{I_k}(g_x) + \frac{\dfrac{5c\rho_n(x)}{2\sigma_n^3(x)} + \dfrac{|(n+1)\,\mu_n(x) - nx|}{\sqrt{2\pi}\sigma_n(x)}}{\sqrt{n}}\,\tilde{h}(x).$$

Now, note that for $0 \leqslant v \leqslant a < 1$, $h(x^+) = f(v^+)$, $h(x^-) = f(v^-)$, $\bar{h}(x) = \bar{f}(v)$, and $\tilde{h}(x) = \tilde{f}(v)$. Furthermore, $V_{I_k}(g_x) = V_{I_k^*}(g_v^*)$ where

$$g_v^*(u) = \begin{cases} f(u) - f(v^+) & \text{if } 1 \geqslant u > v \\ 0 & \text{if } u = v \\ f(u) - f(v^-) & \text{if } 0 \leqslant u < v, \end{cases}$$

$$I_k^* = \left[\frac{v\sqrt{k} - (1-v)}{\sqrt{k} - (1-v)}, \frac{v\sqrt{k} + 1 - v}{\sqrt{k} + 1 - v}\right] \cap [0, 1], \qquad k = 1, 2, ...,$$

and $I_0^* = [0, 1]$. Hence,

$$|P_n(f, v) - \bar{f}(v)| \leqslant \frac{2K_{n+1}(v/(1-v)) + 1}{n} \sum_{k=0}^{n+1} V_{I_k^*}(g_v^*) + \frac{\dfrac{5c\rho_n(v/(1-v))}{2\sigma_n^3(v/(1-v)} + \zeta_n(v)}{\sqrt{n}}\,\tilde{f}(v), \tag{5.2}$$

where $\zeta_n(v) = |(n+1)\,\mu_n(v/(1-v)) - nv/(1-v)|/(\sqrt{2\pi}\sigma_n(v/(1-v)))$.

We may remark in passing that if v is a continuity point of f, then asymptotically the above bound is essentially the best possible. To verify this, let

$$p(u) = |u - v| \qquad \text{for } u \in [0, 1], v \in (0, 1).$$

Now $g_v^*(u) = p(u)$, and $\sum_{k=0}^{n} V_{I_k^*}(g_v^*) \leqslant \lambda_1 + \lambda_2\sqrt{n}$ where λ_1, λ_2 are constants. Since, $K_{n+1}(v/(1-v)) = O(1)$, the right-hand side of (5.2) is $O(1/\sqrt{n})$. On the other hand,

$$\begin{aligned} P_n(p, v) &= E\{h(a_{n+1}S_{n+1})\} \\ &= E\left|\frac{S_{n+1}}{n + S_{n+1}} - v\right|, \end{aligned}$$

where $v = x/(1+x)$. And for $\tau = v(1-v)^2$

$$\sqrt{n}\left(\frac{S_{n+1}}{n+S_{n+1}} - v\right) \xrightarrow{\text{weakly}} N(0, \tau),$$

where $N(0, v)$ is a normal random variable with mean 0 and variance v. Also, $\sqrt{n}\,|S_{n+1}/(n+S_{n+1}) - v|$ are uniformly integrable random variables. Consequently,

$$\sqrt{n}\,|P_n(p, v) - p(v)| = E\left|\sqrt{n}\left(\frac{S_{n+1}}{n+S_{n+1}} - v\right)\right|$$
$$\to E|N(0, \tau)| = \sqrt{\frac{2}{\pi}\tau}$$

which verifies the asertion.

5.3. *Bernstein Power Series (Unmodified)*

For the sake of completeness, analogous results are provided for $Z_n(f, v)$. By Theorem 1, for continuous f

$$|Z_n(f, v) - f(v)| = |A_{n+1}(h, x) - h(x)|$$
$$\leqslant \left(1 + K^*_{n+1}\left(\frac{v}{1-v}\right)\right) \omega\left(f, \frac{1}{\sqrt{n}}\right),$$

where

$$K^*_{n+1}(x) = x(1+x)(1-\omega^* - 2\omega^* x) + (n+1)(x(1+x)\,\omega^*)^2.$$

By Theorem 2, for continuous and convex functions f, $n = 3, 4, \ldots$,

$$K_{n-2}(f, v) = A_{n-1}(h, x) \geqslant A_n(h, x) = K_{n-1}(f, v).$$

Finally by Theorem 4, for $f \in BV[0, 1]$,

$$|Z_n(f, v) - \bar{f}(v)| \leqslant \frac{2K^*_{n+1}(v/(1-v)) + 1}{n} \sum_{k=0}^{n+1} V_{I_k^*}(g_v^*) + \frac{\theta_n(v) + \zeta_n^*(v)}{\sqrt{n}} \tilde{f}(v),$$

where $\zeta_n^*(v) = n\,|\mu_n(v/(1-v)) - v/(1-v)|(\sqrt{2\pi}\,\sigma_n(v/(1-v)))$ and

$$\theta_n(v) = \frac{5c\rho_n(v/(1-v))}{2\sigma_n^3(v/(1-v))}.$$

Again, if v is a continuity point of f, then asymptotically the above bound is essentially the best possible.

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